Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 10, 2005*
Accepted: *February 14, 2005*

First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe

Rong-Gen Cai
Institute of Theoretical Physics, Chinese Academy of Sciences
P.O. Box 2735, Beijing 100080, China, and
CASPER, Department of Physics, Baylor University
Waco, TX76798-7316, U.S.A.
E-mail: cairg@itp.ac.cn

Sang Pyo Kim
Department of Physics, Kunsan National University
Kunsan 573-701, Korea, and
Asia Pacific Center for Theoretical Physics
Pohang 790-784, Korea
E-mail: sangkim@kunsan.ac.kr

Abstract: Applying the first law of thermodynamics to the apparent horizon of a Friedmann-Robertson-Walker universe and assuming the geometric entropy given by a quarter of the apparent horizon area, we derive the Friedmann equations describing the dynamics of the universe with any spatial curvature. Using entropy formulae for the static spherically symmetric black hole horizons in Gauss-Bonnet gravity and in more general Lovelock gravity, where the entropy is not proportional to the horizon area, we are also able to obtain the Friedmann equations in each gravity theory. We also discuss some physical implications of our results.

Keywords: Black Holes, Cosmology of Theories beyond the SM.

 http://jhep.sissa.it/archive/papers/jhep022005050/jhep022005050.pdf

Contents

1. **Introduction** 1

2. **Friedmann equations in Einstein gravity** 3

3. **Friedmann equations in Gauss-Bonnet gravity** 7

4. **Friedmann equations in Lovelock gravity** 8

5. **Conclusion and discussion** 10

1. Introduction

Quantum mechanics together with general relativity predicts that a black hole behaves like a black body, emitting thermal radiations, with a temperature proportional to its surface gravity at the black hole horizon and with an entropy proportional to its horizon area [1, 2]. The Hawking temperature and horizon entropy together with the black hole mass obey the first law of thermodynamics [3]. The formulae of black hole entropy and temperature have a certain universality in the sense that the horizon area and surface gravity are purely geometric quantities determined by the spacetime geometry, once Einstein equations determine the spacetime geometry.

Since the discovery of black hole thermodynamics in 1970's, physicists have been speculating that there should be some relation between black hole thermodynamics and Einstein equations. Otherwise, how does general relativity know that the horizon area of black hole is related to its entropy and the surface gravity to its temperature [4]? Indeed, Jacobson [4] was able to derive Einstein equations from the proportionality of entropy to the horizon area together with the fundamental relation $\delta Q = TdS$, assuming the relation holds for all local Rindler causal horizons through each spacetime point. Here δQ and T are the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. On the other hand, Verlinde [5] found that for a radiation dominated Friedmann-Robertson-Walker (FRW) universe, the Friedmann equation can be rewritten in the same form as the Cardy-Verlinde formula, the latter being an entropy formula for a conformal field theory. Note that the radiation can be described by a conformal field theory. Therefore, the entropy formula describing the thermodynamics of radiation in the universe has the same form as that of the Friedmann equation, which describes the dynamics of spacetime. In particular, when the so-called Hubble entropy bound is saturated, these two equations coincide with each other (for more or less a complete list of references on this topic see, for example, [6]). Therefore, Verlinde's observation further indicates some relation between thermodynamics and Einstein equations (see also [7] for a related discussion)..

JHEP02(2005)050

In a four dimensional de Sitter space with radius l, there is a cosmological event horizon. This horizon, like a black hole horizon, is associated with thermodynamic properties [8]: the Hawking temperature T and entropy S,

$$T = \frac{1}{2\pi l}\,, \qquad S = \frac{A}{4G}\,, \tag{1.1}$$

where $A = 4\pi l^2$ is the cosmological horizon area and G is the Newton constant. For an asymptotic de Sitter space such as a Schwarzschild-de Sitter space, there still exists the cosmological horizon, for which the area law of the entropy holds $S = A/4G$, where A denotes the cosmological horizon area, and whose Hawking temperature is given by $T = \kappa/2\pi$, where κ is the surface gravity of the cosmological horizon. Suppose that some matter with energy dE passes through the cosmological horizon, one then has

$$-\,dE = TdS\,. \tag{1.2}$$

It is easy to verify that the cosmological horizon in the Schwarzschild-de Sitter space satisfies the relation (1.2) (see, for example, [9]). Equation (1.2) is just the first law of thermodynamics for the cosmological horizon.

In the slow-roll inflationary model, the spacetime is a quasi-de Sitter one. If the inflation period is followed by a flat universe with radiation and dust matter, as in the standard big bang universe, the cosmological event horizon is absent in such a universe. However, an apparent horizon always exists, where the expansion vanishes. Frolov and Kofman [10] employed the approach proposed by Jacobson [4] to a quasi-de Sitter geometry of inflationary universe, where they calculated the energy flux of a background slow-roll scalar field (inflaton) through the quasi-de Sitter apparent horizon and used the relation (1.2). Although the topology of the local Rindler horizon in ref. [4] is quite different from that of the quasi-de Sitter apparent horizon considered in ref. [10], it was found that this thermodynamic relation reproduces one of the Friedmann equations with the slow-roll scalar field. It is assumed in their derivation that

$$T = \frac{H}{2\pi}\,, \qquad S = \frac{\pi}{GH^2}\,, \tag{1.3}$$

where H is a slowly varying Hubble parameter.

More recently, following refs. [4, 10], Danielsson [11] has been able to obtain the Friedmann equations, by applying the relation $\delta Q = TdS$ to a cosmological horizon to calculate the heat flow through the horizon in an expanding universe in an acceleration phase and assuming the same form for the temperature and entropy of the cosmological horizon as those in eq. (1.3). Furthermore, Bousso [12] has recently considered thermodynamics in the Q-space (quintessence dominated spacetime). Because the equation of state of quintessence is in the range of $-1 < \omega < -1/3$, the universe undergoes an accelerated expansion, and thus the cosmological event horizon exists in the Q-space. However, Bousso argued that a thermodynamic description of the horizon is approximately valid and thus it would not matter much whether one uses the apparent horizon or the event horizon. Indeed, for the Q-space, the apparent horizon radius R_A differs from the event horizon radius R_E only by

JHEP02(2005)050

a small quantity: $R_A/R_E = 1-\epsilon$, where $\epsilon = 3(\omega+1)/2$. Using the relations

$$T = \frac{1}{2\pi R_A}\,, \qquad S = \frac{\pi R_A^2}{G}\,, \tag{1.4}$$

Bousso showed that the first law (1.1) of thermodynamics holds at the apparent horizon of the Q-space. While these authors [10]–[12] dealt with different aspects of the relation between the first law of thermodynamics and Friedmann equations, they considered only a flat FRW universe.

There is a subtlety that the apparent horizon, the Hubble horizon and the cosmological event horizon cannot be distinguished clearly in some cases where one uses eqs. (1.3), (1.4) or more general forms $T = \kappa/2\pi$ and $S = A/4G$. Therefore, one cannot be sure whether the first law holds for one of the cosmological horizon, the Hubble horizon and the apparent horizon or for some of them or for all of them. Further, when the spatial curvature does not vanish, it would be an interesting issue to see whether one can still derive or not the Friedmann equations from the first law of thermodynamics and to check the relation between thermodynamics and Einstein equations in a more general context. In addition, it is well known that the area formula of black hole entropy holds only in Einstein theory, that is, when the action of gravity theory includes only a linear term of scalar curvature. Therefore, it is worthwhile to study whether, given a relation of entropy and horizon area, one can obtain the Friedmann equations in the corresponding gravity theory from the first law of thermodynamics. In this paper we are going to discuss these issues.

This paper is organized as follows. In section 2, we shall review and clarify the discussions on the relation between thermodynamics and Einstein equations in refs. [10]–[12] and extend the relation to the case of a FRW universe with any spatial curvature in arbitrary dimensions. Applying the first law of thermodynamics to the apparent horizon and assuming the proportionality of entropy and horizon area, we shall successfully derive the Friedmann equations for the universe. In section 3, we shall obtain the Friedmann equations in Gauss-Bonnet gravity by employing the entropy formula for static spherically symmetric Gauss-Bonnet black holes. In section 4, we shall discuss a more general case within the Lovelock gravity. Finally, in section 5, we shall discuss physical implications of the relation.

2. Friedmann equations in Einstein gravity

Let us start with an $(n+1)$-dimensional FRW universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_{n-1}^2\right), \tag{2.1}$$

where $d\Omega_{n-1}^2$ denotes the line element of an $(n-1)$-dimensional unit sphere and the spatial curvature constant $k = +1$, 0 and -1 correspond to a closed, flat and open universe, respectively. The metric (2.1) can be rewritten as [13]

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2\,, \tag{2.2}$$

where $\tilde{r} = a(t)r$ and $x^0 = t$, $x^1 = r$ and the 2-dimensional metric $h_{ab} = \mathrm{diag}(-1, a^2/(1-kr^2))$. The dynamical apparent horizon, a marginally trapped surface with vanishing expansion, is determined by the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$. A simple calculation yields the radius of the apparent horizon

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}, \tag{2.3}$$

where H denotes the Hubble parameter, $H \equiv \dot{a}/a$. Here and hereafter the dots will represent derivatives with respect to the cosmic time t in the metric (2.1). It can be seen from (2.3) that when $k = 0$, namely, for a flat universe, the radius $\tilde{r}_A$ of the apparent horizon has the same value as the radius $\tilde{r}_H$ of the Hubble horizon, which is defined as the inverse of the Hubble parameter, that is, $\tilde{r}_H = 1/H$. On the other hand, the cosmological event horizon defined by

$$\tilde{r}_E = a(t)\int_t^\infty \frac{dt}{a(t)}\,, \tag{2.4}$$

exists only for an accelerated expanding universe. As a consequence, for a pure de Sitter universe with $k = 0$, the apparent horizon, the Hubble horizon and the cosmological event horizon have the same constant value $1/H$. Note that though the cosmological event horizon does not always exist for all FRW universes, the apparent horizon and the Hubble horizon always do exist. For a dynamical spacetime, the apparent horizon has been argued to be a causal horizon and is associated with the gravitational entropy and surface gravity [14, 13]. Thus, for our purpose, it would be convenient to employ the apparent horizon and to apply the first law to the apparent horizon.

Following refs. [13, 14], we define the work density by

$$W = -\frac{1}{2}T^{ab}h_{ab}\,, \tag{2.5}$$

and the energy-supply vector by

$$\Psi_a = T_a{}^b\partial_b\tilde{r} + W\partial_a\tilde{r}\,, \tag{2.6}$$

where T^{ab} is the projection of the $(n+1)$-dimensional energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid matter in the FRW universe in the normal direction of the $(n-1)$-sphere. The work density at the apparent horizon should be regarded as the work done by a change of the apparent horizon, while the energy-supply at the horizon is the total energy flow through the apparent horizon. Then it is shown that one has [14, 13]

$$\nabla E = A\Psi + W\nabla V\,, \tag{2.7}$$

where $A = n\Omega_n\tilde{r}^{n-1}$ and $V = \Omega_n\tilde{r}^n$ are the area and volume of an n-dimensional space with radius $\tilde{r}$, $\Omega_n = \pi^{n/2}/\Gamma(n/2+1)$ being the volume of an n-dimensional unit ball, and that the total energy inside the space with radius $\tilde{r}$ is defined by

$$E = \frac{n(n-1)\Omega_n}{16\pi G}\tilde{r}^{n-2}(1 - h^{ab}\partial_a\tilde{r}\partial_b\tilde{r})\,. \tag{2.8}$$

The equation (2.7) is dubbed *the unified first law* [14]. According to thermodynamics, the entropy is associated with heat flow as $\delta Q = TdS$, and the heat flow is related to the change of energy of the given system. As a consequence, the entropy is finally associated with the energy-supply term. The latter can be rewritten as

$$A\Psi = \frac{\kappa}{8\pi G}\nabla A + \tilde{r}^{n-2}\nabla\left(\frac{E}{\tilde{r}^{n-2}}\right), \tag{2.9}$$

where κ is the surface gravity defined as

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b\tilde{r}) . \tag{2.10}$$

On the apparent horizon the last term in eq. (2.9) vanishes, and one can then assign an entropy $S = A/4G$ to the apparent horizon.

Next, we turn to calculating the heat flow δQ through the apparent horizon during an infinitesimal time interval dt. Heat is one of forms of energy. Therefore, the heat flow δQ through the apparent horizon is just the amount of energy crossing it during that time internal dt. That is, $\delta Q = -dE$ is the change of the energy inside the apparent horizon. Suppose that the energy-momentum tensor $T_{\mu\nu}$ of the matter in the universe has the form of a perfect fluid: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, where ρ and p are the energy density and pressure, respectively. We then find the energy-supply vector

$$\Psi_a = \left(-\frac{1}{2}(\rho + p)H\tilde{r}, \frac{1}{2}(\rho + p)a\right) . \tag{2.11}$$

During the time internal dt, we obtain the amount of energy crossing the apparent horizon:

$$-dE \equiv -A\Psi = A(\rho + p)H\tilde{r}_A dt\,, \tag{2.12}$$

where $A = n\Omega_n\tilde{r}_A^{n-1}$ is the area of the apparent horizon. Assuming that the apparent horizon has an associated entropy S and temperature T

$$S = \frac{A}{4G}\,, \qquad T = \frac{1}{2\pi\tilde{r}_A}\,, \tag{2.13}$$

and then using the first law of thermodynamics, $-dE = TdS$, we finally obtain

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p) . \tag{2.14}$$

Equation (2.14) is nothing but one of Friedmann equations describing an $(n+1)$-dimensional FRW universe with the spatial curvature k. Note that in the procedure to get eq. (2.14), we have used

$$\dot{\tilde{r}}_A = -H\tilde{r}_A^3\left(\dot{H} - \frac{k}{a^2}\right) . \tag{2.15}$$

Once the continuity (conservation) equation of the perfect fluid is given,

$$\dot{\rho} + nH(\rho + p) = 0\,, \tag{2.16}$$

we can substitute $H(\rho + p)$ into (2.14), integrate the resulting equation, and finally obtain

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho . \tag{2.17}$$

This is just another Friedmann equation, the time-time component of Einstein equations. Note that in getting the Friedmann equation (2.17), an integration constant has been dropped out. In fact, this integration constant can be regarded as a cosmological constant, which can be incorporated into the energy density ρ as a special component. We thus have obtained the Friedmann equations (2.14) and (2.17) for the FRW universe by applying the first law of thermodynamics to the apparent horizon. A passing remark is that the above procedure to obtain the Friedmann equations from the first law of thermodynamics can still be applied to the inflationary model with a homogenous scalar field (inflaton), $\phi(t)$. The homogenous scalar field obeys the field equation

$$\ddot{\phi}+nH\dot{\phi}+V'(\phi)=0\,. \tag{2.18}$$

Substituting the energy density and pressure

$$\rho=\frac{1}{2}\dot{\phi}^2+V(\phi),\quad p=\frac{1}{2}\dot{\phi}^2-V(\phi)\,, \tag{2.19}$$

into eq. (2.14) and integrating it, we can find the Friedmann equation for the inflationary model

$$H^2+\frac{k}{a^2}=\frac{16\pi G}{n(n-1)}\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right). \tag{2.20}$$

Careful scrutiny of the above procedure reveals that correct derivation of the Friedmann equations heavily depends on the assumption given in eq. (2.13): the entropy is given by a quarter of the apparent horizon area and the temperature is inversely proportional to the apparent horizon radius $\tilde{r}_A$. The proportionality of entropy and the horizon area can be argued by the area formula of black hole entropy and the so-called *unified first law* (2.9). The assumption on the temperature may be justified as follows. A direct calculation of the surface gravity (2.10) at the apparent horizon gives

$$\begin{aligned}\kappa &= -\frac{\tilde{r}_A}{2}\left(\dot{H}+2H^2+\frac{k}{a^2}\right)\\ &= -\frac{1}{\tilde{r}_A}\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right).\end{aligned} \tag{2.21}$$

For the dynamic apparent horizon, one can see that, in determining the surface gravity, one has to know not only the apparent horizon radius and the Hubble parameter, but also the time-dependence of the horizon radius. Note that for a static or stationary black hole, the surface gravity on the black hole horizon is a constant (the zeroth law of black hole thermodynamics [3]). When the black hole mass changes by an infinitesimally small amount, the horizon radius and thereby the Hawking temperature and entropy will accordingly have a small change. However, the differential form, $dM=TdS$, of the first law of black hole thermodynamics tells us that one needs not to consider the corresponding change of temperature in this procedure of applying the first law of thermodynamics. Therefore, when one applies the first law to the apparent horizon to calculate the surface gravity and thereby the temperature and considers an infinitesimal amount of energy crossing the apparent horizon, the apparent horizon radius $\tilde{r}_A$ should be regarded to have a fixed value.

JHEP02(2005)050

In this sense, we have $\kappa \approx -1/\tilde{r}_A$, and thus recover the relation $T \equiv |\kappa|/(2\pi) = 1/(2\pi\tilde{r}_A)$ between the temperature and surface gravity at the apparent horizon. In this way we justify the assumption of the temperature in eq. (2.13). In other words, the first law of thermodynamics may hold only approximately for the dynamical apparent horizon. This point is worth further studying.

In conclusion, employing the assumption (2.13) and the first law (1.2) of thermodynamics to the dynamic apparent horizon, we are able to obtain the Friedmann equations for an $(n+1)$-dimensional FRW universe with any spatial curvature.

3. Friedmann equations in Gauss-Bonnet gravity

In the previous section we have assumed that the apparent horizon has an entropy proportional to its horizon area. This assumption originated from the black hole thermodynamics: the entropy of black hole horizon obeys the so-called area formula [15]. It is well known that the area formula of black hole entropy no longer holds in higher derivative gravity theories. So it would be interesting to see whether, once given a relation between the entropy and horizon area, one can obtain or not the correct Friedamnn equations for a gravity theory by the approach developed in the previous section. In this section, we shall consider a special higher derivative gravity theory−Gauss-Bonnet gravity.

The action of the Gauss-Bonnet gravity can be written down as

$$S = \frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}(R+\alpha R_{GB}) + S_m\,, \tag{3.1}$$

where α is a constant with the dimension [length]^2, $R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}$ is called the Gauss-Bonnet term and S_m denotes the action of matter. The Gauss-Bonnet term naturally appears in the low energy effective action of heterotic string theory. The Gauss-Bonnet term is a topological term in four dimensions, and thus does not have any dynamic effect in those dimensions. The expansion of Gauss-Bonnet gravity around a flat spacetime is ghost free. The Gauss-Bonnet gravity (3.1) is special in the sense that although the action includes higher derivative curvature terms, there are no more than second-order derivative terms of metrics in equations of motion. Varying the action, one has the equations of motion

$$\begin{aligned}8\pi G T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \\ &- \alpha\left(\frac{1}{2}g_{\mu\nu}R_{GB} - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}{}_{\nu} + 4R_{\gamma\delta}R^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}{}^{\gamma\delta\lambda}\right).\end{aligned} \tag{3.2}$$

In the vacuum Gauss-Bonnet gravity with/without a cosmological constant, static black hole solutions have been found and the associated thermodynamics has been discussed (for example, see [16]). A static, spherically symmetric black hole solution has the metric

$$ds^2 = -e^{\lambda(r)}dt^2 + e^{\nu(r)}dr^2 + r^2 d\Omega_{n-1}^2\,,$$

with

$$e^{\lambda(r)} = e^{-\nu(r)} = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 + \frac{64\pi G\tilde{\alpha}M}{n(n-1)\Omega_n r^n}}\right),$$

where $\tilde{\alpha} = (n-2)(n-3)\alpha$ and M is the mass of black hole. The entropy of the black hole has the following form [16]

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{r_+^2}\right), \tag{3.3}$$

where $A = n\Omega_n r_+^{n-1}$ is the horizon area and r_+ represents the horizon radius.

Now we apply the entropy formula (3.3) to the apparent horizon, assuming that the apparent horizon has the same expression for the entropy as eq. (3.3) but replacing the black hole horizon radius r_+ by the apparent horizon radius $\tilde{r}_A$. That is, the apparent horizon is supposed to have the entropy

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{\tilde{r}_A^2}\right), \tag{3.4}$$

with $A = n\Omega_n \tilde{r}_A^{n-1}$ being the apparent horizon area. We further assume that the apparent horizon still has the temperature $T = 1/(2\pi\tilde{r}_A)$. This is true since the Hawking temperature is determined by geometry itself and has nothing to do with gravity theory explicitly. It means that once the geometry is given, the surface gravity and then the Hawking temperature can be determined immediately. Calculating the amount of energy crossing the apparent horizon, which is still given by eq. (2.12), and applying the first law, $-dE = TdS$, we are led to

$$\left(1 + 2\tilde{\alpha}(H^2 + \frac{k}{a^2})\right)\left(\dot{H} - \frac{k}{a^2}\right) = -\frac{8\pi G}{n-1}(\rho + p)\,. \tag{3.5}$$

Furthermore, substituting the continuity equation (2.16) into eq. (3.5) and integrating the equation, we finally obtain

$$H^2 + \frac{k}{a^2} + \tilde{\alpha}\left(H^2 + \frac{k}{a^2}\right)^2 = \frac{16\pi G}{n(n-1)}\rho\,. \tag{3.6}$$

These equations (3.5) and (3.6) are nothing but the Friedmann equations for a FRW universe in the Gauss-Bonnet gravity given in ref. [17], where holographic entropy bounds have been studied.

Thus, given the relation (3.4) between the entropy and the horizon area in the Gauss-Bonnet gravity theory, we have indeed been able to derive the Friedmann equations (3.5) and (3.6) for the Gauss-Bonnet gravity by applying the first law to the apparent horizon.

4. Friedmann equations in Lovelock gravity

In this section we extend the above discussion to a more general case, the so-called Lovelock gravity [18], which is a generalization of the Gauss-Bonnet gravity. The lagrangian of the Lovelock gravity consists of the dimensionally extended Euler densities

$$\mathcal{L} = \sum_{i=0}^{m} c_i \mathcal{L}_i, \tag{4.1}$$

where c_i is an arbitrary constant and $\mathcal{L}_i$ is the Euler density of a $(2i)$-dimensional manifold

$$\mathcal{L}_i = 2^{-i}\delta^{a_1b_1\cdots a_ib_i}_{c_1d_1\cdots c_id_i} R^{c_1d_1}{}_{a_1b_1} \cdots R^{c_id_i}{}_{a_ib_i}\,. \tag{4.2}$$

Here, the generalized delta function is totally antisymmetric in both sets of indices. $\mathcal{L}_0$ is set to one, therefore the constant c_0 is just the cosmological constant. $\mathcal{L}_1$ gives us the usual curvature scalar term. In order for the general relativity to be recovered in the low energy limit, the constant c_1 has to be positive. For simplicity, we can set $c_1 = 1$. $\mathcal{L}_2$ is just the Gauss-Bonnet term discussed in the previous section. Although the lagrangian of the Lovelock gravity contains higher order derivative curvature terms, there are no terms with more than second order derivatives of metric in equations of motion just as in the Gauss-Bonnet gravity. Therefore, in this sense, the Lovelock theory is not a higher derivative gravity theory.

The lagrangian (4.1) looks complicated. However, static spherically symmetric black hole solutions can be found in this theory in the sense that a metric function is determined by solving for a real root of a polynomial equation [19]. More recently, static, non-spherically symmetric black hole solutions have been also found in the Lovelock gravity [20]. The horizons of these black holes can be hypersurfaces with a positive, zero or negative constant scalar curvature. In particular, it has been shown that the entropy of black hole horizon has a simple expression in terms of the horizon radius [20], while the expression for the metric function and causal structure of these black holes could be quite involved. For an $(n+1)$-dimensional static, spherically symmetric black hole with metric [19, 20]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{n-1}^2\,, \tag{4.3}$$

the metric function is given by $f(r) = 1 - r^2 F(r)$, where $F(r)$ is determined by solving for real roots of the following mth-order polynomial equation

$$\sum_{i=0}^{m} \hat{c}_i F^i(r) = \frac{16\pi G M}{n(n-1)\Omega_n r^n}\,. \tag{4.4}$$

Here, M is an integration constant, which is just the black hole mass, and the coefficients $\hat{c}_i$ are given by

$$\begin{aligned} \hat{c}_0 &= \frac{c_0}{n(n-1)}\,, \qquad \hat{c}_1 = 1\,, \\ \hat{c}_i &= c_i \Pi_{j=3}^{2m}(n+1-j) \qquad \text{for} \qquad i > 1\,. \end{aligned} \tag{4.5}$$

In terms of the horizon radius r_+, the black hole entropy has the expression [20]

$$S = \frac{A}{4G}\sum_{i=1}^{m} \frac{i(n-1)}{(n-2i+1)} \hat{c}_i r_+^{2-2i}\,, \tag{4.6}$$

where $A = n\Omega_n r_+^{n-1}$ is the horizon area of the black hole. Note that the cosmological constant term c_0 does not enter the expression for the black hole entropy. This is reasonable since the black hole entropy is determined by the horizon geometry only [15].

Once again, we assume that the apparent horizon of the FRW universe has an entropy of the form (4.6) with the black hole horizon radius r_+ replaced by the apparent horizon radius $\tilde{r}_A$. The temperature of the apparent horizon is still given by $T = 1/(2\pi\tilde{r}_A)$ and the energy crossing the apparent horizon during the time interval dt is given by eq. (2.12).

JHEP02(2005)050

Thus, the first law, $-dE = TdS$, leads us to the equation

$$\sum_{i=1}^{m} i\hat{c}_i \left(H^2 + \frac{k}{a^2}\right)^{i-1} \left(\dot{H} - \frac{k}{a^2}\right) = -\frac{8\pi G}{n-1}(\rho + p)\,. \tag{4.7}$$

Using the continuity equation (2.16) and integrating eq. (4.7), we finally obtain

$$\sum_{i=1}^{m} \hat{c}_i \left(H^2 + \frac{k}{a^2}\right)^{i} = \frac{16\pi G}{n(n-1)}\rho\,. \tag{4.8}$$

Note that here the integration constant (cosmological constant) has been included in the energy density ρ. These two equations (4.7) and (4.8) are just the Friedmann equations for a FRW universe in the Lovelock gravity [21]. When $m = 2$, equations (4.7) and (4.8) reduce to those corresponding equations (3.5) and (3.6)] of the Gauss-Bonnet gravity.

In other words, employing the entropy form (4.6) for a static, spherically symmetric black hole in the Lovelock gravity, we have obtained correctly the Friedmann equations for a FRW universe in the Lovelock gravity through the first law of thermodynamics at the apparent horizon.

5. Conclusion and discussion

Employing the first law of thermodynamics, $-dE = TdS$, to the apparent horizon of a FRW universe with any spatial curvature in arbitrary dimensions, we have derived the Friedmann equations for the universe. Here $-dE$ denotes the amount of energy crossing the apparent horizon during an infinitesimal time interval and $T = 1/(2\pi\tilde{r}_A)$ is the temperature of the apparent horizon $\tilde{r}_A$. In this procedure to obtain the Friedmann equations, the entropy S is assumed to be a quarter of the apparent horizon area. Using the same form for the relation between entropy and horizon area as that of a static, spherical symmetric black hole in Gauss-Bonnet gravity and its generalization, Lovelock gravity, we have also successfully obtained the Friedmann equations in those gravity theories. These results are certainly related to the holographic properties of gravity. It would be of great interest to study further the implications of these observations to the holographic principle.

The first law of thermodynamics is certainly valid for the cosmological event horizon [8, 9]. It is not a priori clear whether it still holds for a dynamic apparent horizon. Our results in turn also indicate that the first law indeed holds at the apparent horizon. Although the cosmological event horizon does not always exist, an interesting question may arise: whether can we still derive the Friedmann equations by applying the first law of thermodynamics to the cosmological event horizon? Assuming that the event horizon has the temperature and entropy given by eq. (1.3), which are exactly correct for a de Sitter universe, and after a naive calculation, we have found that we can derive the Friedmann equations only for a FRW universe with $k = 0$, that is, for a flat universe. This can be understood since $\tilde{r}_A = 1/H$ for $k = 0$ and the assumptions (1.3) and (1.4) thus become equivalent to each other when $k = 0$. The failure to obtain the Friedmann equations may due to the fact that the assumption (1.3) is not suitable for the cosmological event horizon.

JHEP02(2005)050

In other words, the thermodynamic relations in eq. (1.3) may be not correct for the cosmological event horizon. The failure may also be due to another fact that the Friedmann equations describe local properties of spacetimes, while the cosmological event horizon is determined by global properties of spacetimes (see (2.4)). That is, one cannot determine local properties at each point in a spacetime using global properties of the spacetime, of course, although the inverse is certainly true. We should point out that here the apparent horizon is determined locally. This may be just the reason behind the difference between the cosmological event horizon and the apparent horizon. This is a quite interesting issue worthy of a further study.

Given an entropy relation to the apparent horizon area, we have obtained the correct Friedmann equations for a FRW universe in the Gauss-Bonnet gravity and Lovelock gravity. However, we are not sure whether it is always true for any other gravity theory that, assuming a relation between entropy and its horizon area, the Friedmann equations can be obtained in that gravity theory, where the relation of entropy and area arises. For example, for the gravity theory of the form $\mathcal{L} = f(R)$, where f is a function of scalar curvature R, we know that the entropy of a static black hole is $S = Af'(R)|_{r=r_+}/4G$. Can one derive the Friedmann equations for the gravity theory? These issues are currently under investigation.

Acknowledgments

We thank Seoktae Koh for useful discussions. R.G.C. would like to appreciate the warm hospitality of Kunsan National University. The work of R.G.C. was supported in part by a grant from Chinese Academy of Sciences, by NSFC under grants No. 10325525 and No. 90403029, and by the Ministry of Science and Technology of China under grant No. TG1999075401. The work of S.P.K. was support in part by Korea Astronomy Observatory.

References

[1] S.W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199.

[2] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D 7** (1973) 2333.

[3] J.M. Bardeen, B. Carter and S.W. Hawking, *The four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161.

[4] T. Jacobson, *Thermodynamics of space-time: the Einstein equation of state*, *Phys. Rev. Lett.* **75** (1995) 1260 [gr-qc/9504004].

[5] E. Verlinde, *On the holographic principle in a radiation dominated universe*, hep-th/0008140.

[6] R.G. Cai and Y.S. Myung, *Holography in radiation-dominated universe with a positive cosmological constant*, *Phys. Rev.* **D 67** (2003) 124021 [hep-th/0210272];
S. Nojiri, S.D. Odintsov and S. Ogushi, *Friedmann-Robertson-walker brane cosmological equations from the five-dimensional bulk (A)dS black hole*, *Int. J. Mod. Phys.* **A 17** (2002) 4809 [hep-th/0205187].

[7] T. Padmanabhan, *Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes*, *Class. and Quant. Grav.* **19** (2002) 5387 [gr-qc/0204019]; *Gravity and the thermodynamics of horizons*, *Phys. Rept.* **406** (2005) 49 [gr-qc/0311036].

JHEP02(2005)050

[8] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics and particle creation*, *Phys. Rev.* **D 15** (1977) 2738.

[9] R.G. Cai, *Cardy-Verlinde formula and thermodynamics of black holes in de sitter spaces*, *Nucl. Phys.* **B 628** (2002) 375 [hep-th/0112253]; *Cardy-Verlinde formula and asymptotically de Sitter spaces*, *Phys. Lett.* **B 525** (2002) 331 [hep-th/0111093].

[10] A.V. Frolov and L. Kofman, *Inflation and de Sitter thermodynamics*, *JCAP* **05** (2003) 009 [hep-th/0212327].

[11] U.H. Danielsson, *Transplanckian energy production and slow roll inflation*, *Phys. Rev.* **D 71** (2005) 023516 [hep-th/0411172].

[12] R. Bousso, *Cosmology and the S-matrix*, hep-th/0412197.

[13] D. Bak and S.J. Rey, *Cosmic holography*, *Class. and Quant. Grav.* **17** (2000) L83 [hep-th/9902173].

[14] S.A. Hayward, S. Mukohyama and M.C. Ashworth, *Dynamic black-hole entropy*, *Phys. Lett.* **A 256** (1999) 347 [gr-qc/9810006];
S.A. Hayward, *Unified first law of black-hole dynamics and relativistic thermodynamics*, *Class. and Quant. Grav.* **15** (1998) 3147 [gr-qc/9710089].

[15] R.M. Wald, *Black hole entropy in the noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038].

[16] R.C. Myers and J.Z. Simon, *Black hole thermodynamics in lovelock gravity*, *Phys. Rev.* **D 38** (1988) 2434;
R.G. Cai, *Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev.* **D 65** (2002) 084014 [hep-th/0109133];
R.G. Cai and Q. Guo, *Gauss-bonnet black holes in ds spaces*, *Phys. Rev.* **D 69** (2004) 104025 [hep-th/0311020];
R.G. Cai and K.S. Soh, *Topological black holes in the dimensionally continued gravity*, *Phys. Rev.* **D 59** (1999) 044013 [gr-qc/9808067];
T. Clunan, S.F. Ross and D.J. Smith, *On Gauss-Bonnet black hole entropy*, *Class. and Quant. Grav.* **21** (2004) 3447 [gr-qc/0402044].

[17] R.G. Cai and Y.S. Myung, *Holography and entropy bounds in gauss-bonnet gravity*, *Phys. Lett.* **B 559** (2003) 60 [hep-th/0210300].

[18] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498.

[19] R.C. Myers and J.Z. Simon, *Black hole thermodynamics in lovelock gravity*, *Phys. Rev.* **D 38** (1988) 2434;
J.T. Wheeler, *Symmetric solutions to the gauss-bonnet extended Einstein equations*, *Nucl. Phys.* **B 268** (1986) 737;
J.T. Wheeler, *Symmetric solutions to the maximally gauss-bonnet extended Einstein equations*, *Nucl. Phys.* **B 273** (1986) 732.

[20] R.G. Cai, *A note on thermodynamics of black holes in lovelock gravity*, *Phys. Lett.* **B 582** (2004) 237 [hep-th/0311240].

[21] N. Deruelle and L. Farina-Busto, *The Lovelock gravitational field equations in cosmology*, *Phys. Rev.* **D 41** (1990) 3696.